

A Note on Walsh–Hadamard Feedback Systems as Products of Reflections

1. Setup

Consider the discrete-time system

$$x_{t+1} = Ux_t, \quad U = DH$$

where:

- H is the (normalized) Walsh–Hadamard transform
- $D = \text{diag}(\pm 1, \dots, \pm 1)$ is a diagonal sign matrix

Both (H) and (D) are **orthogonal involutions**:

$$H^2 = I, \quad D^2 = I$$

This makes U the product of two reflections.

2. Two reflections = rotation (the core idea)

A classical geometric fact:

The product of two reflections is a rotation (or, more generally, an orthogonal transformation composed of planar rotations).

In 2D:

- Reflect across line L_1 , then L_2
- Result: rotation by twice the angle between them

In higher dimensions:

- Each reflection flips across a hyperplane
- Their product decomposes into **independent rotations in orthogonal 2D subspaces**

This is a standard result in linear algebra and underlies much of the structure of orthogonal groups.

3. Interpreting H and D as reflections

- D: reflects across coordinate hyperplanes (flips signs of selected coordinates)
- H: reflects across a “global” hyperplane defined by the Walsh basis

So $U = DH$ is:

- reflect in Walsh space
- then reflect in coordinate space

That composition is a **rotation-like operator**.

4. Dynamics: motion on invariant planes

Because U is orthogonal:

- All eigenvalues lie on the unit circle
- The space decomposes into invariant subspaces:
 - 1D: eigenvalues $\pm 1 \rightarrow$ fixed or sign-flip directions
 - 2D: complex conjugate eigenpairs $\rightarrow$ rotations

So the system evolves as:

- independent rotations in multiple 2D planes
- all happening simultaneously

This is the canonical structure of orthogonal dynamics.

5. Why the feedback matters

Without feedback:

- H is just a transform (static)

With feedback:

- iteration turns structure into **dynamics**

Without D:

- $H^2 = I \rightarrow$ trivial 2-cycle

With D:

- symmetry is broken
- nontrivial rotation angles emerge
- longer orbits appear

So D controls the “angles between reflections,” hence the dynamics.

6. Spectral viewpoint (the real lever)

Let $U = DH$. Then:

- eigenvalues: $e^{\pm i\theta_k}$
- each k corresponds to a rotation plane

So:

- dynamics = superposition of rotations at different frequencies
- periodicity depends on whether angles are rational multiples of π

Implications:

- special D $\rightarrow$ short cycles
 - generic D $\rightarrow$ long quasi-periodic orbits
-

7. Change of basis perspective

Using that $H^{-1} = H$:

$$HUH = HD$$

So the system is conjugate to:

- apply sign flips in one basis
- mix in another

This duality is central in:

- Boolean Fourier analysis
 - fast transforms
 - structured random operators
-

8. Relation to discrete harmonic analysis

The Walsh–Hadamard transform diagonalizes functions on the Boolean cube $(\mathbb{Z}_2)^n$.

- D: multiplication by a ± 1 function
- H: change to frequency domain

So U alternates between:

- “position space” (coordinates)
- “frequency space” (Walsh basis)

This is analogous to alternating:

- multiplication and Fourier transform
-

9. What kind of system is this?

Mathematically, it is:

- a linear, norm-preserving dynamical system
- generated by a product of reflections
- fully determined by its spectrum

It is:

- structured
- reversible
- integrable

It is not:

- chaotic (in the linear sense)
 - dissipative
-

10. Deeper connections (for further exploration)

For readers who want to go further, this construction touches on:

- reflection-generated groups and Coxeter theory
 - spectral analysis of structured orthogonal matrices
 - randomized linear algebra (fast mixing transforms)
 - quantum-style circuits (Hadamard + phase layers)
 - operators of the form “multiply then transform”
-

11. Key takeaway

The essential insight is simple but powerful:

Iterating a Walsh–Hadamard transform with sign modulation is not “just a transform”—it is a rotation generated by two reflections.

Everything else—periodicity, mixing behavior, structure—flows from that.
